Energy-efficient train timetables
The two-train separation problem on level track

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- reduce energy costs by up to 20%
- improve on-time arrivals by 10%
- reduce braking by up to 30%
Driving Strategies

- Power
- Hold
- Coast
- Brake

http://scg.ml.unisa.edu.au/dcb.svg
Optimal control of a single train

Equation of motion: \[
\frac{dv}{dt} = \frac{p}{v} - q - r(v) + g(x)
\]

Controls: Tractive power: \(0 \leq p(x) \leq P\)  \(\text{Braking force: } 0 \leq q(x) \leq Q\)
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Problem: Find controls \( p, q \) and an associated speed profile \( v(x) \) with \( v(0) = v(X) = 0 \) and \( t(x) \leq T \) that minimises energy.
If a train travels from A to B, then it consumes less fuel than any other strategy from the standpoint of the optimal control. Generally, it is impossible to have a hold strategy that is best, although it may be the most cost-effective form of control it will compete with in such a way that the cost is minimized. In essence the optimal strategy is identical to the unlimited strategy given by a set of functions of power, coast, and brake. It is possible to incorporate speed limits into the problem with the additional speed constraints if speed cannot be maintained or decreased without braking. Thus such a set will be the unique solution for each power-hold-brake, power-hold-coast-brake, power-coast-brake, and power-brake as shown in Fig. 1. Fig. 3. Optimal control strategies for uphill at speed $V$ and with terminal phase will be interrupted by one or more power-hold-coast-brake, and hence that the control set is non-empty. A feasible strategy can be constructed whenever $\exists$ then this completes within a prescribed time $T$. To pose a meaningful optimal control must ensure that the speed follows the bounds as closely as possible.
What we know about single train control

- On level track the optimal strategy is *power-speedhold-coast-brake*. 
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\[ \Delta_p x(0, V) \]
\[ \Delta_p t(0, V) \]
\[ \Delta_p J(0, V) \]
What we know about single train control

\[ \Delta_p x(0, V) \quad \Delta_h x(a, b, V) \]
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What we know about single train control

\[ \Delta_p x(0, V) \quad \Delta_h x(a, b, V) \quad \Delta c x(V, U) \]
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What we know about single train control

- On level track the optimal strategy is *power-speedhold-coast-brake*.
- As the hold speed increases the journey time decreases.
- There is a unique hold speed for each given journey time.
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- There is a unique hold speed for each given journey time.

We can also find the optimal strategy for:

- Piecewise-constant and continuously varying gradients
- Steep gradients
- Speed limits
What happens with multiple trains?
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That rarely happens!

In practice, trains are often slowed or stopped at signals for safe separation. Instead, trains should be regulated so that they are less likely to encounter restrictive signals.
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So, we want to determine optimal driving strategies for a fleet of trains.
The two-train separation problem

**Problem:** Find driving strategies for two trains travelling on the same track in the same direction subject to given journey times so that an adequate separation is maintained and so that the total energy consumption is minimised.
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In general, the overall strategy may be better if neither train follows the single-train optimal profile.
Maintaining safe separation

- Do not allow two trains on the same track section at the same time.
Maintaining safe separation

- Do not allow two trains on the same track section at the same time.
- Specify intermediate signal times.

A signal time is:
- the latest exit time for the leading train
- the earliest entry for the following train
Outline of a solution procedure

1. Find optimal journeys with prescribed intermediate signal times

2. Find optimal intermediate times
1a. An optimal journey for the leading train

If the optimal unrestricted journey does not leave a particular section before the required time then it must go faster on the first part of the journey.
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If the optimal unrestricted journey does not leave a particular section before the required time then it must go faster on the first part of the journey.

A strategy of optimal type may use different hold speeds for different sections, but as the journey progresses the hold speeds will decrease.

- **power-hold-coast**
  - first section
- **coast-hold-coast**
  - for \((n - 2)\) sections
- **coast-hold-coast-brake**
  - last section
1b. An optimal journey for the following train

If the optimal unrestricted journey enters a particular section before the required time then it must go slower on the first part of the journey.

A strategy of optimal type may use different hold speeds for different sections, but as the journey progresses the hold speeds will increase.

\[
\begin{align*}
\text{power-hold-power} & \quad \text{for } (n - 1) \text{ sections} \\
\text{power-hold-coast-brake} & \quad \text{last section}
\end{align*}
\]
1. Optimal journeys with prescribed intermediate signal times
2. Finding optimal intermediate times

- Calculate times taken to traverse sections (using $\Delta p t$, $\Delta h t$, $\Delta c t$, $\Delta b t$)
- Some requirements:

\[
\sum_{i=0}^{s} f_i \leq \sum_{i=0}^{s-1} g_i + \Delta T \quad \forall s = 1, \ldots, n
\]

\[
\sum_{i=0}^{n} f_i \leq T_\ell
\]

\[
\sum_{i=0}^{n} g_i \leq T_f
\]
2. Finding optimal intermediate times

- Form a Lagrangian function:

\[ \mathcal{J} = J_\ell + J_f + \text{constraints} \]

- Compute the partial derivatives of \( \mathcal{J} \):

\[ \frac{\partial f_i}{\partial V_j}, \quad \frac{\partial g_i}{\partial Y_j} \]

- Arrive at a necessary condition for optimality

This condition means we can check if prescribed times are optimal.
Future work

• Implement an automated search for the hold speeds
• Devise an efficient algorithm to optimise the prescribed intermediate times
• Solve the three-train problem (and then many trains)
• Tackle non-level track
• Integrate with scheduling systems (use timing windows rather than timing points)